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On Approximation Properties of Certain Non-convolution Integral Operators

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INTRODUCTION

Let Ω_0 be a non-empty open set in $\mathbb{R}^m$, $\Omega \subset \Omega_0$ be an open subset of Ω_0, which verifies suitable conditions. Let $f \in L^\infty(\Omega_0) \cap BV(\Omega_0)$ (or $f \in L^1(\Omega_0) \cap BV(\Omega_0)$). In this paper we consider sequences of integral operators $T_n f$ defined by

$$(T_n f)(s) = \int_\Omega K(n, s, t) f(t)\, dt, \tag{I}$$

where $K_n(s, t) = K(n, s, t)$ is a "kernel" belonging to special classes $\mathscr{K}$ which are defined by suitable axioms. Particularly, we assume $T_n f \in W^{1,1}(\Omega)$, for every $f \in BV$. The main theorems of this paper give convergence properties of operators $T_n f$ with respect to certain variational functionals. Given a continuous sublinear function $\mathscr{F}: \mathbb{R}^m \to \mathbb{R}_0^+$, we show that the $\mathscr{F}$-variations of $T_n f$ converge to the $\mathscr{F}$-variation of f (here, by $\mathscr{F}$-variation we mean the measure studied in [23] for the case $\mathscr{F}(p) = |p|$, and in [21] for the general case). Moreover, since the Serrin variational Integral $I_S[\psi, f, \Omega]$, with the integrand $\psi = \psi(p)$ of "area" type, is a suitable $\mathscr{F}$-variation of a $(n+1)$-dimensional vector measure, (see [21]), we show (Theorem 2) that

$$I_S[\psi, T_n f, \Omega] \to I_S[\psi, f, \Omega]. \tag{II}$$

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Then we point out some interesting consequences of this result. For example, (II) implies convergence in length, and in area for the operators $T_n f$. Moreover, by using a result of [4], by (II) we deduce also that grad $T_n f \to$ grad f in measure on Ω, where grad f denotes the "essential" gradient of f (see [23, 28]), that is, the "regular" part of derivative measure of $f \in BV$. We want to mark out that for the special case of length and area in the Cesari sense (see [15] and next developments [20, 30, 9–12, 6]) similar results have been proved by C. Vinti [31], using a different approach. Finally, using a theorem of [5] we may obtain a "weighted" extension of the previous results and so convergence in "weighted" length and area (for these concepts see [9, 10, 5–7]).

1. A Class of Kernels

Let $\Omega \subset \mathbb{R}^m$ be a non-empty open set, $\mathscr{B}(\Omega)$ be the Borel σ-field of Ω. We shall denote by $C_c^k(\Omega)$, $0 \leqslant k \leqslant +\infty$, the class of all C^k-functions with compact support in Ω, and by λ the Lebesgue measure on $\mathscr{B}(\Omega)$. Let $\mathscr{F}: \mathbb{R}^m \to \mathbb{R}_0^+$ be a sublinear, continuous function, that is $\mathscr{F}$ satisfies the following conditions:

(i) $\mathscr{F}(p+q) \leqslant \mathscr{F}(p) + \mathscr{F}(q)$, $p, q \in \mathbb{R}^m$

(ii) $\mathscr{F}(\alpha p) = \alpha \mathscr{F}(p)$, $\alpha \in \mathbb{R}_0^+$, $p \in \mathbb{R}^m$

(iii) $\mathscr{F}(p) \leqslant C\,|p|$, for every $p \in \mathbb{R}^m$ (C is the Lipschitz constant of $\mathscr{F}$).

We denote now by $\mathscr{K}_{\mathscr{F}}(\Omega)$ the class of all functions $K: \mathbb{N} \times \Omega \times \Omega \to \mathbb{R}_0^+$ such that $K(n, \cdot, \cdot)$ is $\mathscr{B}(\Omega) \otimes \mathscr{B}(\Omega)$-measurable, for each $n \in \mathbb{N}$ and such that the following conditions hold:

(k.1) For every $n \in \mathbb{N}$ the function $(s, t) \to K(n, s, t)$ is separately summable in Ω with respect to s and t and there is a sequence $\{a_n\}$ such that $a_n \to 0$ and

$$\int_\Omega K(n, s, t)\, ds = 1 + a_n, \qquad \text{for every} \quad t \in \Omega. \tag{1}$$

(k.2) For every $n \in \mathbb{N}$, the function $H_n(s) \equiv \|K(n, s, \cdot)\|_{L^1(\Omega)}$ is locally λ-summable on Ω.

(k.3) The integral operator

$$(T_n f)(s) = \int_\Omega K(n, s, t)\, f(t)\, dt, \qquad f \in L^\infty(\Omega) \tag{2}$$

is "regularizing," that is, $T_n f \in W^{1,1}(\Omega)$ for every $f \in L^\infty(\Omega)$.

(k.4) For every $f \in L^{\infty}(\Omega)$ and $\varphi \in C_c^{\infty}(\Omega)$, we have

$$\lim_{n \to +\infty} \int \varphi(s)(T_n f)(s)\, ds = \int \varphi(s) f(s)\, ds. \tag{3}$$

(k.5) For every $f \in W^{1,\infty}(\Omega)$, we have

$$\mathscr{F}(\operatorname{grad} T_n f(s)) \leqslant \int_\Omega K(n, s, t)\, \mathscr{F}(\operatorname{grad} f(t))\, dt, \qquad \lambda\text{-a.e., } s \in \Omega. \tag{4}$$

Remarks. (a) If Ω is bounded, condition (k.2) is an easy consequence of (k.1). Indeed, for every compact $S \subset \Omega$ we have

$$\int_S \|K(n, s, \cdot)\|_{L^1(\Omega)}\, ds = \int_S \left\{ \int_\Omega K(n, s, t)\, dt \right\} ds$$

$$= \int_\Omega \left\{ \int_S K(n, s, t)\, ds \right\} dt \leqslant (1 + a_n)\, \lambda(\Omega).$$

For further utilizations of (k.2) (or similar conditions) see [22].

(b) Condition (k.4) expresses an approximation property of the operator $T_n f$, which is satisfied by a large class of integral operators; for example, convolution operators [14, 27], moment kernels [2, 18].

(c) Condition (k.5) is similar to that used by C. Vinti [31] with $\mathscr{F}(p) = |p|$, and connects the gradient of the "mean" $T_n f$ with the "mean" of the gradient of f. In the special case of convolution operators with regular kernels this condition is always verified with $\mathscr{F}(p) = |p|$.

In the following we shall consider, besides the class $\mathscr{K}_{\mathscr{F}}(\Omega)$, also the class $\mathscr{K}^*_{\mathscr{F}}(\Omega)$ of functions $K\colon \mathbb{N} \times \Omega \times \Omega \to \mathbb{R}_0^+$ such that $K(n, \cdot, \cdot)$ is $\mathscr{B}(\Omega) \otimes \mathscr{B}(\Omega)$-measurable for every $n \in \mathbb{N}$, and the following conditions hold:

(k.1)′ For every $n \in \mathbb{N}$, the function $s \to K(n, s, t)$ is summable on Ω, the function $t \to K(n, s, t)$ is $L^{\infty}(\Omega)$ and (1) holds.

(k.3)′ The operator $T_n f$ defined on $L^1(\Omega)$ by (2) is regularizing, that is, $T_n f \in W^{1,1}(\Omega)$, for every $f \in L^1(\Omega)$.

(k.4)′ For every $f \in L^1(\Omega)$, (3) holds.

(k.5)′ For every $f \in W^{1,1}(\Omega)$, (4) holds.

2. The Goffman–Serrin Integral

We denote by $\mathcal{M}^m(\Omega)$ the class of vector measures on $\mathcal{B}(\Omega)$, $\mu: \mathcal{B}(\Omega) \to \mathbb{R}^m$, such that $|\mu|(\Omega) < +\infty$. A function $f \in L^1_{\text{loc}}(\Omega)$ is said to be $BV(\Omega)$ if there is a (vector) measure $\mu_f \in \mathcal{M}^m(\Omega)$ such that

$$\int \varphi \, d\mu_f = -\int (\operatorname{grad} \varphi) f \, d\lambda \tag{5}$$

for every $\varphi \in C_c^\infty(\Omega)$.

For properties of BV functions, see [15, 16, 23, 28, 19]. We write also, $f \in BV(\Omega)$.

If $\mathcal{F}: \mathbb{R}^m \to \mathbb{R}_0^+$ is a sublinear continuous function, we associate to μ_f the positive scalar measure (see [21])

$$\mathcal{F}\mu_f(E) = \sup \sum_{i=1}^{N} \mathcal{F}(\mu_f(E_i)), \qquad E \in \mathcal{B}(\Omega),$$

where the supremum is taken over all finite Borel partitions $E = \bigcup E_i$ of E. This measure has many properties (see [21]). We point out the following semicontinuity property (see [21, 3]); we first premise a definition: a sequence $\{\mu_j\}_j \subset \mathcal{M}^m(\Omega)$ converges weakly to $\mu \in \mathcal{M}^m(\Omega)$ on Ω, if

$$\lim_{j \to +\infty} \int \varphi \, d\mu^j = \int \varphi \, d\mu, \qquad \text{for every} \quad \varphi \in C_c^\infty(\Omega).$$

We shall denote such convergence by $\mu^j \rightharpoonup \mu[\Omega]$. Then it is proved that if $\mu^j \rightharpoonup \mu[\Omega]$, we have

$$\varliminf_{n \to \infty} \mathcal{F}\mu^j(G) \geqslant \mathcal{F}\mu(G), \qquad \text{for every open} \quad G \subseteq \Omega. \tag{6}$$

3. Approximation of $\mathcal{F}\mu_f$

(I) *Case* $f \in L^\infty(\Omega)$

Let $\Omega_0 \subset \mathbb{R}^m$ be a non-empty open set and let Ω be an open set such that $\Omega \subset\subset \Omega_0$, that is, $\Omega \subset \Omega_0$ and $d_\infty(\Omega, \partial\Omega_0) = \delta > 0$, where $d_\infty(x, y) = \max\{|x_i - y_i|; i = 1, ..., n\}$. We shall assume that $f \in L^\infty(\Omega_0) \cap BV(\Omega_0)$. For every sufficiently small $\varepsilon > 0$, we define the "integral mean" f_ε of f on Ω, by setting

$$f^\varepsilon(s) = (2\varepsilon)^{-m} \int_{Q(s,\varepsilon)} f(\xi) \, d\xi, \qquad s \in \Omega, \tag{7}$$

where

$$Q(s, \varepsilon) = \prod_{j=1}^{m} (s_j - \varepsilon, s_j + \varepsilon), \; s = (s_1, ..., s_m) \in \Omega.$$

It is well known that $f^\varepsilon \in W^{1,1}_{\mathrm{loc}}(\Omega)$ and moreover, $f^\varepsilon \in L^\infty(\Omega)$, since $f \in L^\infty(\Omega)$. We have also (see [14, 27, 30]) $f^\varepsilon \to f$ in $L^1_{\mathrm{loc}}(\Omega)$ and $f^\varepsilon(t) \to f(t)$ for every $t \in L_f$, where L_f is the Lebesgue set of f, and so $f^\varepsilon \to f$ a.e. $[\lambda]$ on Ω. Finally

$$|\mathrm{grad}\, f^\varepsilon(s)| = \left| \frac{1}{(2\varepsilon)^m} \int_{Q(s,\varepsilon)} d\mu_f(\xi) \right| \leqslant \frac{|\mu_f|\,(Q(s, \varepsilon))}{(2\varepsilon)^m}$$
$$\leqslant \frac{1}{(2\varepsilon)^m} |\mu_f|\,(\Omega_0) < +\infty.$$

Thus $f^\varepsilon \in W^{1,\infty}(\Omega)$. We now prove some lemmas.

LEMMA 1. *Let* $\mathscr{F}: \mathbb{R}^m \to \mathbb{R}^+_0$ *be a continuous sublinear function,* $K \in \mathscr{K}_{\mathscr{F}}(\Omega)$, *and* $f \in L^\infty(\Omega_0)$.
Then we have

$$\lim_{\varepsilon \to 0^+} \int_\Omega \varphi T_n f^\varepsilon \, ds = \int_\Omega \varphi T_n f \, ds, \tag{8}$$

for every $\varphi \in C^0_c(\Omega)$.

Proof. We first prove that $T_n f^\varepsilon \to T_n f$ in $L^1_{\mathrm{loc}}(\Omega)$, $\varepsilon \to 0$. For every sufficiently small $\varepsilon > 0$, we have

$$|(T_n f^\varepsilon)(s) - (T_n f)(s)| \leqslant \int_\Omega K(n, s, t)\, |f^\varepsilon(t) - f(t)| \, dt.$$

Now for every $n \in \mathbb{N}$ and $s \in \Omega$

$$K(n, s, t)\, |f^\varepsilon(t) - f(t)| \leqslant 2\, \|f\|_\infty \, K(n, s, t)$$

for each $t \in \Omega$ and moreover $K(n, s, t)\, |f^\varepsilon(t) - f(t)|$ goes to 0, for $\varepsilon \to 0$. Hence, by applying (k.1) and the dominated convergence theorem, we deduce

$$\lim_{\varepsilon \to 0} \int_\Omega K(n, s, t)\, |f^\varepsilon(t) - f(t)| \, dt = 0,$$

for every $s \in \Omega$, $n \in \mathbb{N}$. But from the inequality

$$\int_\Omega K(n, s, t)\, |f^\varepsilon(t) - f(t)| \, dt \leqslant 2\, \|f\|_\infty \, \|K(n, s, \cdot)\|_{L^1(\Omega)}$$

and (k.2), the assertion follows. So, if $\varphi \in C_c^0(\Omega)$, setting $S = \operatorname{supp} \varphi$, we have

$$\int_S |\varphi(s)| \, |(T_n f^\varepsilon)(s) - (T_n f)(s)| \, ds$$

$$\leqslant \|\varphi\|_\infty \int_S |(T_n f^\varepsilon)(s) - (T_n f)(s)| \, ds$$

and so the lemma is proved.

Remark. We remark that in the previous lemma we have only used properties (k.1) and (k.2), so it is not necessary that $K \in \mathscr{K}_{\mathscr{F}}(\Omega)$.

For every $K \in \mathscr{K}_{\mathscr{F}}(\Omega)$, let us define the following measures

$$\sigma_n(E) = \int_E \operatorname{grad}(T_n f)(s) \, ds; \qquad v_n^\varepsilon(E) = \int_E \operatorname{grad}(T_n f^\varepsilon)(s) \, ds,$$

where $f \in L^\infty(\Omega_0)$, $E \in \mathscr{B}(\Omega)$, $\Omega \subset\subset \Omega_0$.
Then we have the following:

LEMMA 2. *Let* $K \in \mathscr{K}_{\mathscr{F}}(\Omega)$, $f \in L^\infty(\Omega_0)$. *Then* σ_n, v_n^ε *satisfy the properties*:

(i) $\sigma_n, v_n^\varepsilon \in \mathscr{M}^m(\Omega)$

(ii) $v_n^\varepsilon \rightharpoonup \sigma_n[\Omega]$,

for every $n \in \mathbb{N}$.

Proof. (i) is a direct consequence of (k.3). Thus we prove only (ii). For fixed $n \in \mathbb{N}$, $\varphi \in C_c^\infty(\Omega)$, by (k.3) we have

$$\int \varphi(s) \, dv_n^\varepsilon(s) = \int \varphi(s)(\operatorname{grad} T_n f^\varepsilon)(s) \, ds$$

$$= -\int (\operatorname{grad} \varphi)(s)(T_n f^\varepsilon)(s) \, ds.$$

As the components of grad φ are functions in $C_c^\infty(\Omega)$, applying Lemma 1, we have

$$\lim_{\varepsilon \to 0} \int \varphi \, dv_n^\varepsilon = -\int (\operatorname{grad} \varphi)(T_n f)(s) \, ds$$

$$= \int \varphi(\operatorname{grad} T_n f)(s) \, ds = \int \varphi \, d\sigma_n.$$

Remark. Let $\mathscr{F}$ be a continuous sublinear function on $\mathbb{R}^m$. Since the measures σ_n, ν_n^ε are absolutely continuous with respect to λ, applying Theorem 2 in [21] we have

$$\mathscr{F}\nu_n^\varepsilon(E) = \int_E \mathscr{F}(\text{grad } T_n f^\varepsilon)\, ds, \qquad \mathscr{F}\sigma_n(E) = \int_E \mathscr{F}(\text{grad } T_n f)\, ds,$$

for every $E \in \mathscr{B}(\Omega)$. We are ready to prove the main theorem of this section.

THEOREM 1. *Let $f \in L^\infty(\Omega_0) \cap BV(\Omega_0)$ and let μ_f be the distributional derivative of f. Let $\Omega \subset\subset \Omega_0$ be an open set such that $|\mu_f|\,(\partial\Omega) = 0$ (here $\partial\Omega$ denotes the boundary of Ω). Then, if $K \in \mathscr{K}_{\mathscr{F}}(\Omega)$ we have*

$$\lim_{n\to\infty} \int_\Omega \mathscr{F}(\text{grad } T_n f)\, ds = \mathscr{F}\mu_f(\Omega). \tag{9}$$

Proof. We first prove that $\sigma_n \rightharpoonup \mu_f[\Omega]$. In order to do that, let $\varphi \in C_c^\infty(\Omega)$ be fixed. We have

$$\int \varphi\, d\sigma_n = \int \varphi(\text{grad } T_n f)\, ds = -\int (\text{grad } \varphi)(T_n f)\, ds.$$

By (k.4), taking into account of the fact that $D_i\varphi \in C_c^\infty(\Omega)$, we obtain

$$\lim_{n\to\infty} \int \varphi\, d\sigma_n = -\int (\text{grad } \varphi)\, f\, ds = \int \varphi\, d\mu_f,$$

that is, $\sigma_n \rightharpoonup \mu_f[\Omega]$. Now by the semicontinuity theorem in [3] we obtain

$$\underline{\lim}_{n\to\infty} \int_\Omega \mathscr{F}(\text{grad } T_n f)\, ds \geqslant \mathscr{F}\mu_f(\Omega). \tag{10}$$

Thus it is sufficient to prove that

$$\overline{\lim}_{n\to\infty} \int_\Omega \mathscr{F}(\text{grad } T_n f)\, ds \leqslant \mathscr{F}\mu_f(\Omega). \tag{11}$$

To this end, note that by Lemma 2, and by the semicontinuity property of $\mathscr{F}\mu_f$, we have

$$\underline{\lim}_{\varepsilon\to 0} \int_\Omega \mathscr{F}(\text{grad } T_n f^\varepsilon)\, ds \geqslant \int_\Omega \mathscr{F}(\text{grad } T_n f)\, ds.$$

Since $f^{\varepsilon} \in W^{1,\infty}(\Omega)$, we can apply (k.5) in order to obtain, for each $\varepsilon > 0$,

$$\begin{aligned}
&\int_{\Omega} \mathscr{F}(\operatorname{grad} T_n f^{\varepsilon})\, ds \\
&\quad \leqslant \int_{\Omega} \left\{ \int_{\Omega} K(n, s, t)\, \mathscr{F}(\operatorname{grad} f^{\varepsilon})\, dt \right\} ds \\
&\quad = \int_{\Omega} \mathscr{F}(\operatorname{grad} f^{\varepsilon}) \left\{ \int_{\Omega} K(n, s, t)\, ds \right\} dt = (1 + a_n) \int_{\Omega} \mathscr{F}(\operatorname{grad} f^{\varepsilon})\, dt.
\end{aligned}$$

Now, setting $\Omega^{\varepsilon} = \bigcup_{t \in \Omega} Q(t, \varepsilon)$, and by applying Theorem 1 in [21] we have

$$\begin{aligned}
&\int_{\Omega} \mathscr{F}(\operatorname{grad} f^{\varepsilon})\, dt \\
&\quad = \int_{\Omega} \mathscr{F}\left((2\varepsilon)^{-m} \int_{Q(t,\varepsilon)} d\mu_f(\xi)\right) dt \\
&\quad \leqslant \int_{\Omega} \left\{ (2\varepsilon)^{-m} \int_{Q(t,\varepsilon)} d\mathscr{F}\mu_f(\xi) \right\} dt \\
&\quad \leqslant \int_{\Omega^{\varepsilon}} \left\{ (2\varepsilon)^{-m} \int_{Q(\xi,\varepsilon)} dt \right\} d\mathscr{F}\mu_f(\xi) = \mathscr{F}\mu_f(\Omega^{\varepsilon}).
\end{aligned}$$

Therefore,

$$\int_{\Omega} \mathscr{F}(\operatorname{grad} T_n f^{\varepsilon})\, ds \leqslant (1 + a_n)\, \mathscr{F}\mu_f(\Omega^{\varepsilon}).$$

Thus, as $|\mu_f|\,(\partial\Omega) = 0$, we obtain, for $\varepsilon \to 0$,

$$\begin{aligned}
&(1 + a_n)\, \mathscr{F}\mu_f(\bar{\Omega}) = (1 + a_n)\, \mathscr{F}\mu_f(\Omega) \\
&\qquad \geqslant \varliminf_{\varepsilon \to 0} \int_{\Omega} \mathscr{F}(\operatorname{grad} T_n f^{\varepsilon})\, ds \geqslant \int_{\Omega} \mathscr{F}(\operatorname{grad} T_n f)\, ds.
\end{aligned}$$

Consequently, as $n \to \infty$, we obtain (11) and so (9).

(II) *Case* $f \in L^1(\Omega)$

Let $\Omega_0 \subset \mathbb{R}^m$ be a non-empty open set and let $\Omega \subset \Omega_0$ be an open set such that $\Omega \subset\subset \Omega_0$. Let $f \in L^1(\Omega_0) \cap BV(\Omega_0)$ and $K \in \mathscr{K}^*_{\mathscr{F}}(\Omega)$. We shall prove a result which is analogous to Theorem 1. The proof is based on the following variant of Lemma 1, which, in this setting, gives a stronger result.

LEMMA 1′. *Let $\mathscr{F}: \mathbb{R}^m \to \mathbb{R}_0^+$ be a continuous sublinear function. If $K \in \mathscr{K}^*_{\mathscr{F}}(\Omega)$, for each $n \in \mathbb{N}$ and $f \in L^1(\Omega_0)$ it results*

$$\lim_{\varepsilon \to 0^+} \|T_n f^\varepsilon - T_n f\|_{L^1(\Omega)} = 0.$$

Proof. We have

$$\|T_n f^\varepsilon - T_n f\|_{L^1(\Omega)} \leqslant \int_\Omega \|K(n, \cdot, t)[f^\varepsilon(t) - f(t)]\|_{L^1(\Omega)}\, dt,$$

and by (1),

$$\|K(n, \cdot, t)[f^\varepsilon(t) - f(t)]\|_{L^1(\Omega)} = \int_\Omega K(n, s, t)\, |f^\varepsilon(t) - f(t)|\, ds$$
$$= (1 + a_n)\, |f^\varepsilon(t) - f(t)|$$

and hence

$$\|T_n f^\varepsilon - T_n f\|_{L^1(\Omega)} \leqslant (1 + a_n) \int_\Omega |f^\varepsilon(t) - f(t)|\, dt.$$

Since $f \in L^1(\Omega)$, we have $\|f^\varepsilon - f\|_{L^1(\Omega)} \to 0$ (see, e.g., [27]), and so the assertion follows.

Now, by similar arguments, we prove:

THEOREM 1′. *Let $K \in \mathscr{K}^*_{\mathscr{F}}(\Omega)$ and let $f \in L^1(\Omega_0) \cap BV(\Omega_0)$; then we have*

$$\lim_{n \to \infty} \int_\Omega \mathscr{F}(\text{grad } T_n f)\, ds = \mathscr{F}\mu_f(\Omega),$$

where $\Omega \subset\subset \Omega_0$ and $|\mu_f|\,(\partial\Omega) = 0$.

Remarks. (a) In the previous theorems, we may assume that the regularization properties of the operators $T_n f$, are verified only for functions in $BV(\Omega_0) \cap L^\infty(\Omega_0)$ or $[BV(\Omega_0) \cap L^1(\Omega_0)]$.

(b) The "integral means" employed in the proofs of Theorems 1 and 1′ may be replaced by "mollifiers" operators (see, e.g., [19]). Thus we may assume that inequality (4) holds only for $C^\infty \cap W^{1,\infty}$ functions. In this case, we can replace (K.5) with the following condition (K.6). There exist $r \in \mathbb{N}$, $\{b_n\}$, $b_n \to 1$, such that, for every $f \in C^\infty \cap W^{1,\infty}(\Omega)$, we have

$$\mathscr{F}(\text{grad } T_n f)(s) \leqslant b_n \int_\Omega K(n + r, s, t)\, \mathscr{F}(\text{grad} f(t))\, dt.$$

We note that for the moment kernel, (K.6) is verified with $r = 1$ and $b_n = (n + 1)/(n + 2)$.

4. Applications

Let $\Omega_0 \subset \mathbb{R}^m$ be a non-empty open set. Let $f \in L^1(\Omega_0) \cap BV(\Omega_0)$ and let $\Omega \subset\subset \Omega_0$ be an open set. Then $|\mu_f|(\Omega) < +\infty$ represents the total variation of f on Ω. Suppose that $\mathscr{F}(p) = |p|$, $p \in \mathbb{R}^m$. In this case, Theorem 1′ has an important geometric meaning: that is, the "means" $T_n f$ of f converge in variation to f; so, if $E \in \mathscr{B}(\Omega)$ and φ_E is the characteristic function of E, the number $|\mu_{\varphi_E}|(\Omega)$ is the perimeter of E (see [17, 24, 30, 19]) and if $\varphi_E \in BV(\Omega_0)$, E is said to be "Cacioppoli set." Hence the previous theorems provide convergence in perimeter of the means $T_n f$ of f. Similar results, for $\mathscr{F}(p) = |p|$ and $\Omega_0 = \mathbb{R}^m$, have been stated by C. Vinti in [31], using a different approach. Our aim is now to prove a convergence theorem for $T_n f$ with respect to the Serrin variational Integral [29], with an integrand $\psi = \psi(p)$ which depends only on the gradient of f. Well, let $\psi: \mathbb{R}^m \to \mathbb{R}_0^+$ be a convex function such that the limit $\psi^*(p) \equiv \lim_{t \to 0^+} t\psi(p/t)$ exists and is finite. Then it is possible to define a continuous sublinear function $\mathscr{G}: \mathbb{R}^{m+1} \to \mathbb{R}_0^+$ on putting

$$\mathscr{G}(p, t) = t\psi(p/t), \qquad t > 0; \ \mathscr{G}(p, 0) = \psi^*(p).$$

Let $\Omega_0 \subset \mathbb{R}^m$ be a bounded open set, $f \in L^1(\Omega_0) \cap BV(\Omega_0)$ and put $\overline{\mu_f} = (\mu_f, \lambda)$. Then $\overline{\mu_f} \in \mathscr{M}^{m+1}(\Omega_0)$, by boundness of Ω_0. Let us define for every $E \in \mathscr{B}(\Omega_0)$,

$$\psi\mu_f(E) = \mathscr{G}\overline{\mu_f}(E).$$

Moreover, we set $\overline{\sigma_n} = (\sigma_n, \lambda)$, $\bar{v}_n^\varepsilon = (v_n^\varepsilon, \lambda)$ where σ_n and v_n^ε are the measures defined in Section 3. Finally, we denote by $\mathscr{K}_\psi^*(\Omega)$, Ω open set with $\Omega \subset\subset \Omega_0$, the class of all functions $K: \mathbb{N} \times \Omega \times \Omega \to \mathbb{R}_0^+$ that satisfy the same properties of the class $\mathscr{K}_{\mathscr{F}}^*(\Omega)$ with ψ instead of $\mathscr{F}$. We now prove the following:

THEOREM 2. *Let $\Omega \subset\subset \Omega_0$ be an open set such that $|\overline{\mu_f}|(\partial\Omega) = 0$, and let $K \in \mathscr{K}_\psi^*(\Omega)$. If $f \in L^1(\Omega_0) \cap BV(\Omega_0)$, we have*

$$\lim_{n \to \infty} \int_\Omega \psi(\text{grad } T_n f)\, ds = \psi\mu_f(\Omega) = \mathscr{G}\overline{\mu_f}(\Omega). \tag{12}$$

Proof. Since $v_n^\varepsilon \rightharpoonup \sigma_n[\Omega]$ we have also $\overline{v_n^\varepsilon} \rightharpoonup \overline{\sigma_n}[\Omega]$, and so

$$\varliminf_{\varepsilon \to 0^+} \int_\Omega \psi(\text{grad } T_n f^\varepsilon)\, ds \geq \int_\Omega \psi(\text{grad } T_n f)\, ds. \tag{13}$$

By (k.5)′ with ψ instead of $\mathscr{F}$, we have

$$\begin{aligned}\int_\Omega \psi(\operatorname{grad} T_n f^\varepsilon)\, ds &\leqslant \int_\Omega \left\{\int_\Omega K(n, s, t)\, \psi(\operatorname{grad} f^\varepsilon)\, dt\right\} ds \\ &= \int_\Omega \psi(\operatorname{grad} f^\varepsilon) \left\{\int_\Omega K(n, s, t)\, ds\right\} dt = (1 + a_n) \int_\Omega \psi(\operatorname{grad} f^\varepsilon)\, dt.\end{aligned}$$

Now,

$$\begin{aligned}\int_\Omega \psi(\operatorname{grad} f^\varepsilon)\, dt &= \int_\Omega \psi\left((2\varepsilon)^{-m} \int_{Q(t,\varepsilon)} d\mu_f(\xi)\right) dt \\ &= \int_\Omega \mathscr{G}\left((2\varepsilon)^{-m} \int_{Q(t,\varepsilon)} d\mu_f(\xi), 1\right) dt \\ &\leqslant \int_\Omega \left\{(2\varepsilon)^{-m} \int_{Q(t,\varepsilon)} d\mathscr{G}\overline{\mu_f}(\xi)\right\} dt \leqslant \mathscr{G}\overline{\mu_f}(\Omega^\varepsilon),\end{aligned}$$

where $\Omega^\varepsilon = \bigcup_{t \in \Omega} Q(t, \varepsilon)$.

As $\varepsilon \to 0^+$, by $|\overline{\mu_f}|\,(\partial\Omega) = 0$ and (13) we obtain

$$(1 + a_n)\, \mathscr{G}\overline{\mu_f}(\Omega) \geqslant \int_\Omega \psi(\operatorname{grad} T_n f)\, ds, \qquad n \in \mathbb{N}.$$

From this, as $n \to +\infty$, we obtain

$$\overline{\lim_{n \to \infty}} \int_\Omega \psi(\operatorname{grad} T_n f)\, ds \leqslant \psi\mu_f(\Omega). \tag{14}$$

Finally, since $\overline{\sigma_n} \rightharpoonup \overline{\mu_f}$ (see Theorem 1), the assertion follows by semicontinuity of $\mathscr{G}\mu_f$.

Remark. The same remarks we have made after Theorem 1′ remain valid also in this setting.

EXAMPLE 1. Let $m = 1$, $\psi(p) = \sqrt{1 + p^2}$. If $f \in L^1(\Omega_0)$, the previous theorem gives convergence in length for functions $T_n f$.

EXAMPLE 2. Let $m = 2$, $\psi(p, q) = \sqrt{1 + p^2 + q^2}$, $f \in L^1(\Omega_0)$. In this case Theorem 2 gives convergence in area for functions $T_n f$.

The following is an interesting consequence of Theorem 2 and Theorem 3.1 in [4].

COROLLARY 1. *Under the assumptions of Theorem 2, if, moreover ψ is positive and strictly convex and verifies the property*

$$\psi^*(p)=0 \Leftrightarrow p=0, \qquad (*)$$

then it results

$$\lim_{n\to+\infty} \psi^*(\operatorname{grad} T_n f - \operatorname{grad} f)=0,$$

in λ-measure on Ω (here grad f *is the "essential gradient" of f).*

For example, if $\psi\colon \mathbb{R}^2 \to \mathbb{R}_0^+$ is the function $\psi(p, q)=\sqrt{1+p^2+q^2}$, by Corollary 1 we deduce

$$\frac{\partial}{\partial x} T_n f(x, y) \to \frac{\partial}{\partial x} f(x, y)$$

$$\frac{\partial}{\partial y} T_n f(x, y) \to \frac{\partial}{\partial y} f(x, y)$$

in λ-measure on Ω (here $(\partial/\partial x) f$, $(\partial/\partial y) f$ are the "essential" partial derivatives of f).

5. INTRODUCTION OF A WEIGHT

Let $\Omega_0 \subset \mathbb{R}^m$ be a non-empty open set, $f \in L^1(\Omega_0) \cap BV(\Omega_0)$, $\Omega \subset\subset \Omega_0$ and μ_f be the derivative measure of f. Let $\nu\colon \mathscr{B}(\Omega_0) \to \mathbb{R}_0^+$ be a (scalar) measure such that

$$\nu(E)=\int_E g\, d\lambda, \qquad E \in \mathscr{B}(\Omega_0),$$

where $g \in C^0(\Omega_0)$ and $0<\lambda \leqslant g(s) \leqslant \Lambda$, for every $s \in \Omega_0$. We now prove the following

COROLLARY 2. *Let $f \in L^1(\Omega_0) \cap BV(\Omega_0)$, $\Omega \subset\subset \Omega_0$, $|\mu_f|\,(\partial\Omega)=0$. Let $K \in \mathscr{K}^*_{\mathscr{F}}(\Omega)$, where $\mathscr{F}$ is a continuous sublinear functional such that $\mathscr{F}(p)=0 \Leftrightarrow p=0$. Then*

$$\lim_{n\to+\infty} \int_\Omega \mathscr{F}(\operatorname{grad} T_n f)\, g(s)\, ds = \mathscr{F}\tilde{\mu}_f(\Omega),$$

where $\tilde{\mu}_f\colon \mathscr{B}(\Omega_0) \to \mathbb{R}^m$ is the (vector) measure defined by

$$\tilde{\mu}_f(E)=\int_E g(s)\, d\mu_f(s), \qquad E \in \mathscr{B}(\Omega_0).$$

Proof. The assertion easily follows by $\sigma_n \rightharpoonup \mu_f$, Theorem 1′, and Theorem 2 of [5].

Remarks. (a) Corollary 2 implies a theorem of convergence in weighted perimeter and variation (see [6, 9]) for the operators $T_n f$, $f \in L^1$. Clearly, a similar result holds for $f \in L^r$. Moreover, with analogous reasoning we may prove a weighted version of Theorem 2, by putting $\tilde{\bar{\mu}}_f(E) = \int_E g\, \overline{d\mu_f}$, $E \in \mathscr{B}(\Omega_0)$, and thus we obtain convergence in weighted area and length for operators $T_n f$.

(b) We note that all the results remain valid if the (1) of (k.1) is replaced by the more general assumption

$$\int_\Omega K(n, s, t)\, ds = g_n(t),$$

where g_n is a sequence of non-negative functions such that

$$\lim_{n \to +\infty} \sup_{t \in \Omega} g_n(t) = 1.$$

It is sufficient to put $1 + a_n = \sup_{t \in \Omega} g_n(t)$ in the proofs.

REFERENCES

1. E. BAIADA, Convergence in variation and convergence in length, *Atti Sem. Mat. Fis. Univ. Modena* **15** (1966), 98–102.
2. F. BARBIERI, Approssimazione mediante nuclei momento, *Atti Sem. Mat. Fis. Univ. Modena* **32** (1983), 308–328.
3. C. BARDARO, La disuguaglianza generalizzata di Steiner per funzionali sublineari su misure, *Atti Sem. Mat. Fis. Univ. Modena* **30** (1981), 114–130.
4. C. BARDARO, Alcuni teoremi di convergenza per l'integrale multiplo del Calcolo delle Variazioni, *Atti Sem. Mat. Fis. Univ. Modena* **31** (1982), 302–324.
5. C. BARDARO, Indipendenza dal peso della convergenza di funzionali sublineari su misure vettoriali ed applicazioni, *Rend. Ist. Mat. Univ. Trieste* **19** (1987), 44–63.
6. C. BARDARO AND G. VINTI, Perimetro e variazione generalizzata rispetto ad una misura in $\mathbb{R}^2$, *Atti Sem. Mat. Fis. Univ. Modena* **35** (1987), 173–190.
7. F. BIAGINI AND M. RAGNI, Sull'area generalizzata con peso vettoriale, *Atti Sem. Mat. Fis. Univ. Modena*, in press.
8. M. BONI, Sull'approssimazione dell'Integrale multiplo del Calcolo delle Variazioni, *Atti Sem. Mat. Fis. Univ. Modena* **20** (1971), 187–211.
9. M. BONI, Variazione generalizzata con peso e quasi additività, *Atti Sem. Mat. Fis. Univ. Modena* **25** (1976), 195–210.
10. M. BONI AND A. SALVADORI, Convergenza in lunghezza e variazione in senso generalizzato con peso, *Atti Sem. Mat. Fis. Univ. Modena* **26** (1977), 49–69.
11. P. BRANDI AND A. SALVADORI, Sull'area generalizzata, *Atti Sem. Mat. Fis. Univ. Modena* **28** (1979), 33–62.

12. P. BRANDI AND A. SALVADORI, On convergence in area in the generalized sense, *J. Math. Anal. Appl.* **92** (1983), 119–138.
13. H. BREZIS, "Analyse Fonctionelle: Théorie et Application," Masson, Paris, 1983.
14. P. L. BUTZER AND R. J. NESSEL, "Fourier Analysis and Approximation," Academic Press, New York/London, 1971.
15. L. CESARI, Sulle funzioni a variazione limitata, *Ann. Scuola Norm. Sup. Pisa* **5** (1936), 229–313.
16. L. CESARI, "Optimization Theory and Applications," Springer-Verlag, New York/Berlin, 1983.
17. E. DE GIORGI, Su una teoria generale della misura $(r-1)$-dimensionale in uno spazio ad r dimensioni, *Ann. Mat. Pura Appl. (4)* **36** (1954), 191–213.
18. C. FIOCCHI, Nuclei momento due-dimensionali e convergenza in area, *Atti Sem. Mat. Fis. Univ. Modena* **33** (1984), 291–312.
19. E. GIUSTI, Minimal surfaces and functions of bounded variation, *in* "Monographs in Mathematics," Vol. 80, Birkhäuser, Basel, 1984.
20. C. GOFFMAN, Lower semicontinuity and area functionals. I. The non parametric case, *Rend. Circ. Mat. Palermo* **2** (1953), 203–234.
21. C. GOFFMAN AND J. SERRIN, Sublinear functions of measures and variational integral, *Duke Math. J.* **31** (1964), 159–178.
22. M. A. KRASNOSELSKII *et al.*, "Integral Operators on Spaces of Summable Functions," Noordhoff, Groningen, 1976.
23. K. KRICKEBERG, Distributionen, Funktionen beschrankter Variation und Lebesguescher Inhalt nichtparametrischer Flachen, *Ann. Mat. Pura Appl.* **44** (1957), 105–133.
24. M. MIRANDA, Distribuzioni aventi derivate misure. Insiemi di perimetro localmente finito, *Ann. Scuola Norm. Sup. Pisa (3)* **19** (1964), 27–56.
25. T. RADÒ, Length and area, *Ann. Math. Soc. Colloq. Publ.* **30** (1948).
26. T. RADÒ AND P. REICHELDERFER, Convergence in length and convergence in area, *Duke Math. J.* **9** (1942), 527–565.
27. C. SADOSKY, "Interpolation of Operators and Singular Integrals," Dekker, NY, 1979.
28. J. SERRIN, On differentiability of functions of several variables, *Arch. Rational Mech. Anal.* **7** (1961), 358–372.
29. J. SERRIN, On the definition and properties of certain variational integrals, *Trans. Amer. Math. Soc.* **101** (1961), 139–167.
30. C. VINTI, Perimetro-Variazione, *Ann. Scuola Norm. Sup. Pisa (3)* **18** (1964), 201–231.
31. C. VINTI, Sull'approssimazione in perimetro ed in area, *Atti Sem. Mat. Fis. Univ. Modena* **13** (1964), 187–197.

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